Chapter 1 Additional Questions

8) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \text{ converges if, and only if, } \sigma > 1.$$

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^{\sigma}} \text{ converges if, and only if, } \sigma > 1. \tag{23}$$

$$\sum_{n=3}^{\infty} \frac{1}{n \log n\,(\log\log n)^{\sigma}} \text{ converges if, and only if, } \sigma > 1.$$

Can you see a pattern?

9) i Using the observation that $\log 1 = 0$ improve the result of Question 4i to

$$N \log N - N + 2 - \log 2 \le \sum_{1 \le n \le N} \log n \le (N+1)\log N - N + 2 - 2\log 2.$$

ii. Deduce that for such N,

$$\frac{e^2}{2}\left(\frac{N}{e}\right)^N \le N! < \frac{N}{2}\left(\frac{e^2}{2}\left(\frac{N}{e}\right)^N\right). \tag{24}$$

This is an increase on the lower bound of (20) by a factor of $e/2 \approx 1.359..$ and a decrease on the upper bound by a factor of $e/4 \approx 0.679.....$ Alternatively the improvement can be seen in that the lower and upper bounds in (24) differ by a factor of $N/2$ whilst in the earlier (20) the factor is N. Can we do better and have a factor that does not depend on N? See a later Problem Sheet.

10) Recall from notes the definition of the set

$$\mathcal{N} = \{n : p|n \Rightarrow p \le N\}.$$

Then unique factorisation of integers justifies, for *real* $s = \sigma > 1$, the last equality in

$$\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \ge \sum_{n \in \mathcal{N}} \frac{1}{n^{\sigma}} = \prod_{p \le N}\left(1 - \frac{1}{p^{\sigma}}\right)^{-1}. \tag{25}$$

This is a rather convoluted way of showing that $\zeta(\sigma) > 0$ for $\sigma > 1$ since this *finite* product can not be zero (to be zero one of the factors would have to be zero). Yet can (25) be generalised to *complex* s so that we can conclude that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$?

In the lectures we prove that

$$\left|\zeta(s) - \prod_{p \le N}\left(1 - \frac{1}{p^s}\right)^{-1}\right| \le \sum_{n \notin \mathcal{N}} \frac{1}{n^\sigma} \le \sum_{n \ge N+1} \frac{1}{n^\sigma} \le \frac{1}{(\sigma - 1)\, N^{\sigma - 1}}. \qquad (26)$$

and deduced that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (27)$$

for $\operatorname{Re} s > 1$. From this we see, on multiplying both sides of (27) by a *finite* number of terms that, for any $N > 1$,

$$\prod_{p \le N}\left(1 - \frac{1}{p^s}\right)\zeta(s) = \prod_{p > N}\left(1 - \frac{1}{p^s}\right)^{-1}. \qquad (28)$$

i) Prove that for any $M > N$,

$$\left|\prod_{N < p \le M}\left(1 - \frac{1}{p^s}\right)^{-1} - 1\right| \le \frac{1}{(\sigma - 1)\, N^{\sigma - 1}}.$$

Hint Write the product as a sum and use the same ideas that gave the bound (26) .

ii) Deduce that given $s : \operatorname{Re} s > 1$ there exists $N = N(s) > 1$ such that

$$\left|\prod_{p \le N}\left(1 - \frac{1}{p^s}\right)\zeta(s) - 1\right| \le \frac{1}{2}.$$

Hint Take a limit over M in Part i.

iii) Deduce

$$|\zeta(s)| > \frac{1}{2}\left|\prod_{p \le N}\left(1 - \frac{1}{p^s}\right)^{-1}\right|.$$

Hint Perhaps use the triangle inequality in the form $|a| > |b| - |a - b|$.

This is our generalisation of (25). The N depends on s but for a given s it is finite and this finite product is never zero and so $|\zeta(s)| > 0$, i.e. $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$.

11) i) By looking at an integral justify

$$\log\left(\frac{1}{1-x}\right) < 2x$$

for $0 \leq x < 1/2$.

ii) Use this to prove a weaker form of Theorem 1.4,

$$\sum_{p \leq N} \frac{1}{p} > \frac{1}{2} \log\log(N+1).$$

(It may be weaker, but the proof is shorter.)

Hint Part i gives

$$2\frac{1}{p} > \log\left(1-\frac{1}{p}\right)^{-1} \quad \text{so} \quad 2\sum_{p\leq N} \frac{1}{p} > \log \sum_{n \in \mathcal{N}} \frac{1}{n}.$$

Why? This latter sum over n is seen in the proof of Theorem 1.4 and so follow the steps found there.

12) A function f is *convex* on $[a, b]$ iff

$$f(a + t(b-a)) \leq f(a) + t\,(f(b) - f(a))$$

for all $0 \leq t \leq 1$. That is, the graph for f between $x = a$ and $x = b$ lies **below** the chord joining the points $(a, f(a))$ and $(b, f(b))$

i) Prove that if f is convex then

$$\int_a^b f(t)\,dt \leq \frac{1}{2}(b-a)\,(f(b) + f(a)).$$

ii) Prove that $1/t$ is concave on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Deduce that

$$\log\left(\frac{1}{1-x}\right) - x \leq \frac{x^2}{2\,(1-x)}$$

for $0 < x < 1$.

What change does this lead to in Theorem 1.4?

13) For $\sigma > 1$ the Riemann zeta function converges absolutely and so

$$\zeta(\sigma) \geq \sum_{n=1}^{N} \frac{1}{n^\sigma}, \tag{29}$$

for all $N \geq 1$, while from Theorem 1.11 we have

$$\zeta(\sigma) = \prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-1} \tag{30}$$

for such σ. Assume that there are only *finitely* many primes and use (30) and (31) to obtain a contradiction.

Hint For each prime p the factor of the Euler Product is continuous, i.e.

$$\lim_{\sigma \to \sigma_0} \left(1 - \frac{1}{p^\sigma}\right)^{-1} = \left(1 - \frac{1}{p^{\sigma_0}}\right)^{-1},$$

for all $\sigma_0 \in \mathbb{R}$. From second year analysis the *finite* product (or sum) of functions continuous at a point, is continuous at that point, i.e. for two functions f and g if $\lim_{\sigma \to \sigma_0} f(\sigma) = f(\sigma_0)$ and $\lim_{\sigma \to \sigma_0} g(\sigma) = g(\sigma_0)$ then

$$\lim_{\sigma \to \sigma_0} (f(\sigma) + g(\sigma)) = f(\sigma_0) + g(\sigma_0)$$

and

$$\lim_{\sigma \to \sigma_0} (f(\sigma)\, g(\sigma)) = f(\sigma_0)\, g(\sigma_0).$$

By repeated application these results hold for finitely many terms in the sum or product. Use these facts in your solution.

For comparison the *infinite* sum or product of functions continuous at a point, are **not** necessarily continuous at that point.

14) Prove that Theorem 1.4, or more precisely Question 1 above, implies

$$\sum_p \frac{1}{p^\sigma} \geq e^{-1} \left(\log\left(\frac{1}{\sigma - 1}\right) - 1\right),$$

for $1 < \sigma < 1 + 1/\log 3$.

A weaker version of Theorem 1.13.

Hint Given σ, truncate the sum at x, to be chosen. Get the sum into a one of summing $1/p$, not $1/p^\sigma$. BUT, to simply say

$$\frac{1}{p^\sigma} \geq \frac{1}{p}$$

throws away too much information. Instead write

$$\frac{1}{p^\sigma} = \frac{1}{p}\left(\frac{1}{p}\right)^{\sigma-1} \geq \frac{1}{p}\left(\frac{1}{x}\right)^{\sigma-1},$$

since $p \leq x$. Finally use Question 1 and then choose x in terms of σ.